

Quantum spectral curve for (q, t) -matrix model

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Abstract

We derive quantum spectral curve equation for (q, t) -matrix model, which turns out to be a certain difference equation. We show that in Nekrasov–Shatashvili limit this equation reproduces the Baxter TQ equation for the quantum XXZ spin chain. This chain is spectral dual to the Seiberg–Witten integrable system associated with the AGT dual gauge theory.

1 Introduction

Quantum generalizations of classical spectral curves have attracted much attention recently. They appear in a variety of different contexts in mathematical physics, some of them interrelated. In topological strings quantum spectral curves are the equations solved by the topological recursion procedure [1]. In $2d$ CFT they provide the differential equation for conformal block with a degenerate field insertion [2]. In $\mathcal{N} = 2$ gauge theories quantum spectral curves describe Nekrasov partition functions with extra surface operators [3], [4]. In quantum integrable systems they appear as Baxter TQ equations or universal difference operators [5]. Finally in knot theory similar equations provide recurrence relations for colored HOMFLY polynomials in different representations [6].

In this short letter we derive the quantum spectral curve for the (q, t) -matrix model. It is related to all of the fields mentioned above. Firstly, the (q, t) -matrix models (or the q -deformed β -ensemble) provide the Dotsenko–Fateev representation for the q -deformed Virasoro conformal blocks [7]. Due to the AGT correspondence [8], these blocks are related to five dimensional gauge theories [9]. In the Nekrasov–Shatashvili limit [10] the resulting equation turns into the Baxter TQ equation for the quantum XXZ spin chain. Finally, quantum spectral curve can be thought of as an equation for the partition function of refined topological strings on toric Calabi-Yau three-fold with a Lagrangian brane attached to one of the legs of the toric diagram [11]. This partition function can be computed using (refined) topological vertex technique [12] and in simple cases is related to certain knot invariants [20].

In sec. 2.1 we warm up with the well known example of the ordinary β -ensemble. In sec. 2.2 we proceed to our main object: the (q, t) -matrix model. In sec. 3 we look at the

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NS limit of the curve and obtain the Baxter TQ equation for the XXZ spin chain. We also clarify its relation to the Seiberg–Witten integrable system which in this case is given by *another* XXZ spin chain. Some comments on the obtained result are given in sec. 4.

2 Dotsenko–Fateev representation and quantum spectral curves

2.1 Virasoro conformal block and β -ensemble

Let us start with the usual Virasoro conformal blocks with a single degenerate field insertion V_{degen} . The differential equation for this conformal block is obtained from the null-vector condition $(L_{-1}^2 + aL_{-2})V_{\text{degen}}(z) = 0$. We will now translate this statement into the language of β -ensembles.

Conformal blocks can be represented as correlators in a free field theory with additional screening operators. This produces the Dotsenko–Fateev integral representation [13], [14]:

$$\begin{aligned} \mathcal{B}_{\text{Vir}}(\{\Lambda_a\}, \{\Delta_a\}) &= \mathcal{B}_{\text{Heis}}^{-1}(\{\Lambda_a\}, \{\Delta_a\}) \int_{\{\mathcal{C}\}} d^N x \Delta^{2\beta}(x) \prod_{i=1}^N \prod_{a=1}^K (\Lambda_a - x_i)^{v_a} = \\ &= \mathcal{B}_{\text{Heis}}^{-1}(\{\Lambda_a\}, \{\Delta_a\}) I_{\text{DF}}(\{\Lambda_a\}, \{\Delta_a\}, \{\mathcal{C}\}), \quad (1) \end{aligned}$$

where $\{\mathcal{C}\}$ is a collection of contours, which determines the intermediate dimensions in the conformal block, $\Delta^{2\beta}(x) = \prod_{i \neq j} (x_i - x_j)^\beta$, Λ_a are the positions of the primary fields, and v_a are Liouville momenta of the primaries. Notice a factor $\mathcal{B}_{\text{Heis}}^{-1}$, which is the conformal block of the Heisenberg algebra. It is also known as the “ $U(1)$ part” in the AGT literature and is given by a simple explicit expression in Λ_a and v_a . From now on we will work exclusively with the integral itself and not concern ourselves with the Heisenberg part.

To derive the differential equation one needs a primary field with $v_a = 1$. We denote the position of this field by z . Consider the following integral of a total derivative:

$$\begin{aligned} 0 &= \int_{\{\mathcal{C}\}} d^N x \sum_{i=1}^N \frac{\partial}{\partial x_i} \left[\frac{1}{z - x_i} \Delta^{2\beta}(x) \prod_{k=1}^N (z - x_k) \prod_{i=1}^N \prod_{a=1}^K (\Lambda_a - x_i)^{v_a} \right] = \\ &= \int_{\{\mathcal{C}\}} d^N x \prod_{k=1}^N (z - x_k) \sum_{i=1}^N \frac{1}{z - x_i} \frac{\partial}{\partial x_i} \left[\Delta^{2\beta}(x) \prod_{i=1}^N \prod_{a=1}^K (\Lambda_a - x_i)^{v_a} \right] = \\ &= \int_{\{\mathcal{C}\}} d^N x \prod_{k=1}^N (z - x_k) \sum_{i=1}^N \frac{1}{z - x_i} \left(\sum_{j \neq i} \frac{2\beta}{x_i - x_j} - \sum_{a=1}^K \frac{v_a}{\Lambda_a - x_i} \right) \left[\Delta^{2\beta}(x) \prod_{i=1}^N \prod_{a=1}^K (\Lambda_a - x_i)^{v_a} \right] = \\ &= \left(\beta \partial_z^2 - \sum_{a=1}^K \frac{v_a}{z - \Lambda_a} \partial_z + \sum_{a=1}^K \frac{1}{z - \Lambda_a} \partial_{\Lambda_a} \right) I_{\text{DF}}(\{z, \Lambda_a\}, \{\Delta_a\}, \{\mathcal{C}\}) \quad (2) \end{aligned}$$

We thus obtain a well-known second order differential equation in z for the conformal block with a single degenerate field. We now move on to the q -deformed case.

2.2 q -Virasoro conformal block and (q, t) -matrix model

In the q -deformed case the DF integral becomes Jackson q -integral¹:

$$I_{\text{DF}}(\{\Lambda_a\}, \{v_a\}, \{\mathcal{C}\}) = \int_{\{\mathcal{C}\}} d_q^N x \Delta^{(q,t)}(x) \prod_{i=1}^N \prod_{a=1}^K \prod_{k=0}^{v_a-1} (\Lambda_a - q^k x_i), \quad (3)$$

where $\{\mathcal{C}\}$ is a collection of contours, $\Delta^{(q,t)}(x) = \prod_{k=0}^{\beta} \prod_{i \neq j} (x_i - q^k x_j)$, $t = q^\beta$ and v_a, Λ_a are the parameters².

Naturally the quantum spectral curve should now be a difference equation. The degenerate field insertion in the q -deformed case stays the same: it amounts to adding $\prod_{i=1}^N (z - x_i)$ into the integral. The derivatives ∂_z , though, should be somehow deformed to q -derivatives $(1 - q^{z\partial_z})$.

As in the previous section, consider the q -integral of a total q -derivative:

$$\begin{aligned} 0 &= \int_{\{\mathcal{C}\}} d_q^N x \sum_{i=1}^N \frac{1}{x_i} (q^{x_i \partial_{x_i}} - 1) \left[\frac{x_i \prod_{a=1}^K \left(\Lambda_a - \frac{x_i}{q} \right)}{z - x_i} \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j} \times \right. \\ &\quad \left. \times \Delta^{(q,t)}(x) \prod_{m=1}^N (z - x_m) \prod_{a=1}^K \prod_{k=0}^{v_a-1} (\Lambda_a - q^k x_m) \right] = \\ &= \int_{\{\mathcal{C}\}} d_q^N x \prod_{k=1}^N (z - x_k) \sum_{i=1}^N \frac{1}{z - x_i} (q^{x_i \partial_{x_i}} - 1) \left[\prod_{a=1}^K \left(\Lambda_a - \frac{x_i}{q} \right) \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j} \times \right. \\ &\quad \left. \times \Delta^{(q,t)}(x) \prod_{m=1}^N \prod_{a=1}^K \prod_{k=0}^{v_a-1} (\Lambda_a - q^k x_m) \right] \quad (4) \end{aligned}$$

Notice a nontrivial factor of $\prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j}$, which is inherent to the q -deformed case and does not appear in the limit $q \rightarrow 1$. The shift operator $q^{x_i \partial_{x_i}}$ acts on $\Delta^{(q,t)}(x)$ as follows:

$$q^{x_i \partial_{x_i}} \Delta^{(q,t)}(x) = t^{N-1} \prod_{j \neq i} \frac{(qx_i - x_j)(tx_i - x_j)}{(x_i - x_j)(qx_i - tx_j)} \Delta^{(q,t)}(x). \quad (5)$$

Some factors in Eq. (5) cancel with the additional factor $\prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j}$. One obtains the following identity:

$$\begin{aligned} 0 &= \int_{\{\mathcal{C}\}} d_q^N x \left[\sum_{i=1}^N \frac{qt^{N-1}}{z - x_i} \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} \prod_{a=1}^K (\Lambda_a - q^{v_a} x_i) - \right. \\ &\quad \left. - \sum_{i=1}^N \frac{1}{z - x_i} \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j} \prod_{a=1}^K \left(\Lambda_a - \frac{x_i}{q} \right) \right] \times \\ &\quad \times \Delta^{(q,t)}(x) \prod_{k=1}^N (z - x_k) \prod_{a=1}^K \prod_{k=0}^{v_a-1} (\Lambda_a - q^k x_i). \quad (6) \end{aligned}$$

¹Jackson integral is defined as follows: $\int_0^a f(x) d_q x = (1 - q) \sum_{n \geq 0} a q^n f(q^n a) = \frac{1-q}{1-q^a} a f(a)$.

²We implicitly assume β and v_a to be integer in these expressions, however everything has a well-defined analytic continuation to non-integer values. In particular, the spectral curve equation is valid for any values of the parameters.

Let us write both terms in the square brackets as contour integrals³:

$$\begin{aligned} \sum_{i=1}^N \frac{qt^{N-1}}{z-x_i} \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} \prod_{a=1}^K (\Lambda_a - q^{v_a} x_i) &= \\ &= \frac{qt^{N-1}}{t-1} \oint_{\mathcal{C}_x} \frac{dw}{w} \frac{1}{z-w} \prod_{j=1}^N \frac{tw - x_j}{w - x_j} \prod_{a=1}^K (\Lambda_a - q^{v_a} w), \quad (7) \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^N \frac{1}{z-x_i} \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j} \prod_{a=1}^K \left(\Lambda_a - \frac{x_i}{q} \right) &= \\ &= \frac{1}{1-t} \oint_{\mathcal{C}_x} \frac{dw}{w} \frac{1}{z-w} \prod_{j=1}^N \frac{w - tx_j}{w - x_j} \prod_{a=1}^K \left(\Lambda_a - \frac{w}{q} \right), \quad (8) \end{aligned}$$

where the contour \mathcal{C}_x encircles all the points x_i .

We deform the contour \mathcal{C}_x to \mathcal{C}_∞ — a large contour around zero — so that it gets extra contributions from residues at z and 0 . We will treat the integral over \mathcal{C}_∞ in a moment. Let us first find the residues at points z and 0 :

$$\begin{aligned} \frac{qt^{N-1}}{t-1} \oint_{\mathcal{C}_x} \frac{dw}{w} \frac{1}{z-w} \prod_{j=1}^N \frac{tw - x_j}{w - x_j} \prod_{a=1}^K (\Lambda_a - q^{v_a} w) &= \\ &= -\frac{qt^{N-1}}{z(t-1)} \left[\prod_{a=1}^K \Lambda_a - \prod_{j=1}^N \frac{tz - x_j}{z - x_j} \prod_{a=1}^K (\Lambda_a - q^{v_a} z) \right], \quad (9) \end{aligned}$$

$$\begin{aligned} \frac{1}{1-t} \oint_{\mathcal{C}_x} \frac{dw}{w} \frac{1}{z-w} \prod_{j=1}^N \frac{w - tx_j}{w - x_j} \prod_{a=1}^K \left(\Lambda_a - \frac{w}{q} \right) &= \\ &= \frac{1}{z(t-1)} \left[t^N \prod_{a=1}^K \Lambda_a - \prod_{j=1}^N \frac{z - tx_j}{z - x_j} \prod_a \left(\Lambda_a - \frac{z}{q} \right) \right]. \quad (10) \end{aligned}$$

One can notice that the factors containing x_j in Eqs. (9), (10) can be expressed as shift operators acting on the degenerate field:

$$\prod_{j=1}^N \frac{tz - x_j}{z - x_j} \prod_{j=1}^N (z - x_j) = t^{z\partial_z} \prod_{j=1}^N (z - x_j), \quad (11)$$

$$\prod_{j=1}^N \frac{z - tx_j}{z - x_j} \prod_{j=1}^N (z - x_j) = t^N t^{-z\partial_z} \prod_{j=1}^N (z - x_j). \quad (12)$$

³We include the factor $\frac{1}{2\pi i}$ in the definition of the contour integral.

We now calculate the troublesome residues at infinity:

$$\begin{aligned} \frac{qt^{N-1}}{t-1} \oint_{\mathcal{C}_\infty} \frac{dw}{w} \frac{1}{z-w} \prod_{j=1}^N \frac{tw-x_j}{w-x_j} \prod_{a=1}^K (\Lambda_a - q^{v_a} w) = \\ = \frac{qt^{N-1}}{(t-1)} \frac{1}{(K-1)!} \frac{\partial^{K-1}}{\partial y^{K-1}} \left(\frac{1}{yz-1} \prod_{j=1}^N \frac{t-yx_j}{1-yx_j} \prod_{a=1}^K (\Lambda_a y - q^{v_a}) \right) \Big|_{y=0}, \quad (13) \end{aligned}$$

$$\begin{aligned} \frac{1}{1-t} \oint_{\mathcal{C}_\infty} \frac{dw}{w} \frac{1}{z-w} \prod_{j=1}^N \frac{w-tx_j}{w-x_j} \prod_{a=1}^K \left(\Lambda_a - \frac{w}{q} \right) = \\ = \frac{1}{q^K(1-t)} \frac{1}{(K-1)!} \frac{\partial^{K-1}}{\partial y^{K-1}} \left(\frac{1}{yz-1} \prod_{j=1}^N \frac{1-tyx_j}{1-yx_j} \prod_{a=1}^K (\Lambda_a qy - 1) \right) \Big|_{y=0}, \quad (14) \end{aligned}$$

For lower K the total residue at infinity, which we denote by $R_\infty^{(K)}$, can be found explicitly:

$$\begin{aligned} R_\infty^{(0)} &= 0, \\ R_\infty^{(1)} &= \frac{1}{q(t-1)} (t^{2N-1} q^{v_1+2} - 1), \\ R_\infty^{(2)} &= \frac{1}{q^2(t-1)} \left[z (t^{2N-1} q^{3+v_1+v_2} - 1) + (t-1) (q^{3+v_1+v_2} t^{2N-2} + 1) \sum_{j=1}^N x_j + \right. \\ &\quad \left. + \sum_{a=1}^2 \Lambda_a q^{-v_a} (q^{1+v_a} - q^{3+v_1+v_2} t^{2N-1}) \right] \end{aligned}$$

In general $R_\infty^{(K)}(z)$ is a polynomial in z of degree $K-1$, of the form

$$R_\infty^{(K)}(z) = \frac{t^{2N-1} q^{1+\sum_{a=1}^K v_a} - q^{-K}}{t-1} (-z)^{K-1} + \dots \quad (15)$$

The coefficient in front of z^m contains symmetric polynomials in x_j up to degree $K-m-1$. In the NS limit these symmetric polynomials decouple from the matrix model average and become Hamiltonians of the XXZ spin chain. Before the limit, they can be obtained by acting on $I_{\text{DF}}(z)$ with certain operators in Λ_a . We denote the operator polynomial, which produces $R_\infty^{(K)}(z)$ from $I_{\text{DF}}(z)$ as $\hat{R}_\infty^{(K)}(z)$. However, we will not write out these operators explicitly.

Finally, we obtain the quantum spectral curve for the (q, t) -matrix model:

$$\begin{aligned} \left[-\frac{t^N}{z(t-1)} \left(\left(\frac{q}{t} - 1 \right) \prod_{a=1}^K \Lambda_a - \frac{q}{t} \prod_{a=1}^K (\Lambda_a - q^{v_a} z) t^{z\partial_z} + \prod_{a=1}^K \left(\Lambda_a - \frac{z}{q} \right) t^{-z\partial_z} \right) + \right. \\ \left. + \hat{R}_\infty^{(K)}(z) \right] I_{\text{DF}}(z) = 0. \quad (16) \end{aligned}$$

One usually sets the position of one of the fields in the conformal block to zero, which simply amounts to taking the limit $\Lambda_1 \rightarrow 0$. Then the equation simplifies:

$$\boxed{\left[\frac{t^N}{(t-1)} \left(\frac{q^{v_1+1}}{t} \prod_{a=2}^K (\Lambda_a - q^{v_a} z) t^{z\partial_z} - \frac{1}{q} \prod_{a=2}^K \left(\Lambda_a - \frac{z}{q} \right) t^{-z\partial_z} \right) + \hat{R}_\infty^{(K)}(z) \right] I_{\text{DF}}(z) = 0} \quad (17)$$

3 Nekrasov–Shatashvili limit

In the NS limit the quantum spectral curve can be interpreted as a Baxter equation for a certain integrable system [16] (for a more mathematical description see [3]). In the ordinary Virasoro case this system is just the Hitchin integrable system associated to the Riemann surface (with marked points) on which the conformal block lives. Generalizing this result, we show that the q -deformed conformal block on a sphere satisfies the Baxter TQ equation for the XXZ spin chain.

3.1 Virasoro block and Hitchin system

Again we start with the well-known example of Virasoro conformal block. In the NS limit $\epsilon_2 \rightarrow 0$, $\epsilon_1 = \text{const}^4$ so that $\beta \rightarrow \infty$, and the dimensions of the fields in the conformal block also have to be rescaled with β , so that $v_a = \beta \tilde{v}_a$, $\tilde{v}_a = \text{const}$. The main simplification in the quantum spectral curve equation (2) occurs because the dimension of the degenerate field is fixed and does not scale with β . Therefore, the correlators involving $V_{\text{degen}}(z)$, in particular $I_{\text{DF}}(z)$, behave as $e^{-\beta \mathcal{F} + \mathcal{S}(z) + \dots}$, where the leading factor \mathcal{F} is independent of z . To get rid of the leading factor one can divide by the DF integral $I_{\text{DF},0} \simeq e^{-\beta \mathcal{F}}$ *without* the degenerate field insertion.

Finally, we obtain an equation for the “wave function” $\Psi(z) = \frac{I_{\text{DF}}(\{z, \Lambda_a\})}{I_{\text{DF},0}(\{\Lambda_a\})}$:

$$\left(\partial_z^2 - \sum_{a=1}^K \frac{\tilde{v}_a}{z - \Lambda_a} \partial_z + \sum_{a=1}^K \frac{h_a}{z - \Lambda_a} \right) \Psi(z) = 0 \quad (18)$$

where $h_a = \partial_{\Lambda_a} \mathcal{F}(\{\Lambda_a\})$. This is nothing but the Baxter equation for the \mathfrak{gl}_2 Gaudin model with K marked points which is the relevant Hitchin system in this case. The Baxter equation can be written in the form

$$: \det_{2 \times 2}(\partial_z - L(z)) : \Psi(z) = 0, \quad (19)$$

where $L(z) = \sum_{a=1}^K \frac{A_a}{z - \Lambda_a}$ is the Lax matrix and an appropriate normal ordering is imposed on the operators in the determinant. The parameters of the β -ensemble are interpreted as the Casimir operators and Hamiltonians of the Gaudin model as follows:

$$\tilde{v}_a = \text{tr } A_a, \quad (20)$$

$$h_a = \sum_{b \neq a} \frac{\text{tr } A_a A_b}{\Lambda_a - \Lambda_b}. \quad (21)$$

⁴The parameters ϵ_1 and ϵ_2 enter Nekrasov function on equal footing, so there is an equivalent form of the NS limit $\epsilon_1 \rightarrow 0$, $\epsilon_2 = \text{const}$, which we employ in the next section.

3.2 q -deformed block and XXZ spin chain

In the q -deformed case one has $q = e^{-R_5 \epsilon_1}$, $t = e^{R_5 \epsilon_2}$ and in the NS limit $\epsilon_1 \rightarrow 0$, $\epsilon_2 = \text{const.}$ This choice is related to the convention of the previous section by the symmetry $\epsilon_1 \leftrightarrow \epsilon_2$. One also has to rescale the v_a parameters so that $q^{v_a} = t^{\tilde{v}_a} = \text{const.}$

As in the ordinary Virasoro case, in the NS limit the operator $\hat{R}_\infty^{(K)}(z)$ act on the DF integral without insertion $I_{\text{DF},0}$ and not on the wavefunction $\Psi(z) = \frac{I_{\text{DF}}(\{z, \Lambda_a\})}{I_{\text{DF},0}(\{\Lambda_a\})}$. Thus the operator $\hat{R}_\infty^{(K)}(z)$ is replaced by a polynomial $\frac{t^N}{t-1} P_K(z) = \left(I_{\text{DF},0}^{-1}(\{\Lambda_a\}) \hat{R}_\infty^{(K)}(z) I_{\text{DF},0}(\{\Lambda_a\}) \right)$ and the quantum spectral curve (17) takes the form

$$\left[t^{\tilde{v}_1-1} \prod_{a=2}^K (\Lambda_a - t^{\tilde{v}_a} z) t^{z\partial_z} - \prod_{a=2}^K (\Lambda_a - z) t^{-z\partial_z} + P_{K-1}(z) \right] \Psi(z) = 0 \quad (22)$$

This equation has exactly the same form as the Baxter equation for the XXZ spin chain with $K-1$ spins from $U_q(\mathfrak{gl}_2)$ algebra:

$$: \det_{2 \times 2} (q^{z\partial_z} - T(z)) : Q(z) = 0 \quad (23)$$

where $T(z) = V \prod_{a=2}^K L_a(z)$ is the transfer matrix. More explicitly

$$[\mathfrak{q} K_+(z) q^{z\partial_z} - K_-(z) q^{-z\partial_z} + P_{K-1}(z)] Q(z) = 0, \quad (24)$$

where $K_\pm(z) = \prod_{a=2}^K (z - m_a^\pm)$ and $P_{K-1}(z) = \sum_{a=0}^{K-1} h_a z^{K-a-1}$ are polynomials of degree $K-1$ containing Casimir operators of the spins and the Hamiltonians of the chain respectively and \mathfrak{q} is the parameter of the twist matrix. The matrix model parameters are identified with the spin chain parameters as follows:

$$m_a^+ = \Lambda_a t^{-\tilde{v}_a}, \quad (25)$$

$$m_a^- = \Lambda_a, \quad (26)$$

$$\mathfrak{q} = t^{\sum_{a=1}^K \tilde{v}_a - 1} \quad (27)$$

and one should write q instead of t in Eq. (22).

Let us clarify the relation of the integrable system we have just obtained and the Seiberg–Witten integrable system [15]. Due to the AGT correspondence the q -deformed conformal block we consider is equal to partition function of the five-dimensional $SU(2)^{K-2}$ gauge theory. In turn this gauge theory is associated with the XXZ spin chain with two spins from symmetric representations of $U_q(\mathfrak{gl}_{K-1})$ algebra. This is notably *different* from the spin chain described by (24), which has $K-1$ spins of $U_q(\mathfrak{gl}_2)$.

The resolution of this seeming discrepancy requires the notion of *spectral duality*. There are in fact two equivalent chains with the Baxter equations related by this duality (it can be thought of as just the Fourier transform in the spectral parameter) [17]. One is the Seiberg–Witten integrable system, while the other is the system arising from the AGT dual conformal block.

4 Conclusions and discussion

We have derived the quantum spectral curve for the (q, t) -matrix model and shown that in the NS limit it reduces to the Baxter equation for the XXZ spin chain. Spectral duality connects this spin chain and the corresponding Seiberg–Witten integrable system. This should be compared with the quantum spectral curve for the β -ensemble, which in the NS limit is related to Hitchin system.

There are several directions in which our result can be generalized. Firstly, one can consider the toric q -deformed conformal blocks and the corresponding Dotsenko–Fateev integrals. They should satisfy a difference equation, of the elliptic Ruijsenaars form. It would also be interesting to study elliptic deformations of the Virasoro *algebra*, which should lead to XYZ spin chains and *double elliptic* integrable models [18]. Probably Baxter equations for these cases might clarify the relationship between the spectral duality [17] and p-q duality in integrable models [19]. Recently the connection between $5d$ gauge theories described by the matrix models we have considered here and polynomial knot invariants has been obtained [20]. This direction is also worth investigating in the future.

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